Fourier Law and Non-Isothermal Boundary in the Boltzmann Theory
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Steady Boltzmann Equation

\[ \nu \cdot \nabla_x F = Q(F, F) \]

- \( F(x, \nu) \): density distribution of rarefied gas
- 3D velocity space \( \nu \in \mathbb{R}^3 \)
- \( \Omega \): bounded, connected domain in \( \mathbb{R}^d \) for \( d = 1, 2, 3 \)
- nonlinear Boltzmann operator \( Q(F_1, F_2) \):
  - quadratic, bilinear
  - non-local in \( \nu \in \mathbb{R}^3 \)
  - hard potential \( 0 \leq \gamma \leq 1 \) with angular cut-off
  - collision invariant: \( \int_{\mathbb{R}^3} \{1, \nu, |\nu|^2\} Q(F, F)(\nu) d\nu = 0 \)
- Knudsen number \( \sim 1 \) regime
Non-Isothermal Boundary and Diffusive BC

Wall temperature

\[ \theta(x) = \theta_0 + \delta \vartheta(x) \quad \text{on} \quad x \in \partial \Omega \]

Diffusive boundary condition on \( x \in \partial \Omega, \; n(x) \cdot v < 0 \)

\[ F(x, v) = \mu^\theta(x, v) \int_{n(x) \cdot u > 0} F(x, u)\{n(x) \cdot u\} \, du \]

Wall Maxwellian

\[ \mu^\theta(x, v) = \frac{1}{2\pi\theta(x)^2} \exp \left[ -\frac{|v|^2}{2\theta(x)} \right] \]

with \( \int_{n(x) \cdot v > 0} \mu^\theta(x, v)\{n(x) \cdot v\} \, dv = 0 \)
Purpose of This Work

- Analyze the thermal conduction phenomena in the kinetic regime (Knudsen number $\sim 1$)
Purpose of This Work

- Analyze the thermal conduction phenomena in the kinetic regime (Knudsen number $\sim 1$) when the wall temperature does not oscillate too much!

  \[
  \left| \theta(x) - \theta_0 \right| \ll 1, \quad |\vartheta(x)| \leq 1 \text{ and } \delta \ll 1
  \]

  \[
  F_s \sim \mu \quad \text{Regime}
  \]
Natural Questions and Previous Works

- **Existence, Uniqueness, Non-Negativity for Steady Solution**
  - S.-H.Yu: existence and stability, $\Omega$ is slab (length $\ll 1$), ARMA 2009

- **Regularity (Continuity and Singularity)**
  - Y.Guo: for IBVP, $\Omega$ convex, continuity away from $\gamma_0$: ARMA 2010
  - C.K: for IBVP, $\Omega$ non-convex, singularity formation and propagation: CMP 2011
Dynamical Stability
- C. Villani: polynomial decay in $H^k$, diffusive BC, $\theta \equiv \theta_0$: Mem. AMS 2009
- Y. Guo: $\theta \equiv \theta_0$, $e^{-\lambda t}$—decay in $L^\infty$ to $\mu$: ARMA 2010
- S.-H. Yu: $e^{-\lambda t}$—decay in $L^\infty$ to the steady solution: ARMA 2009

Hydrodynamic Limit
Theorem: Existence, Uniqueness and Non-Negativity

Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$.
For all $M > 0$,
Theorem: Existence, Uniqueness and Non-Negativity

Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$.
For all $M > 0$, there exists $\delta_0 > 0$ such that for $0 < \delta < \delta_0$ in

$$|\theta(x) - \theta_0| \leq \delta, \quad \text{on} \quad x \in \partial \Omega,$$

then there exists a non-negative solution $F_s = M\mu + \sqrt{\mu}f_s \geq 0$ with $\int\int_{\Omega \times \mathbb{R}^3} f_s \sqrt{\mu} = 0$ to the problem

$$\nu \cdot \nabla_x F_s = Q(F_s, F_s), \quad F_s|_{\gamma^-} = \mu^\theta \int_{\gamma^+} F_s d\gamma,$$

where $\theta(x)$ is the temperature.
Theorem: Existence, Uniqueness and Non-Negativity

Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$.

For all $M > 0$, there exists $\delta_0 > 0$ such that for $0 < \delta < \delta_0$ in

$$|\theta(x) - \theta_0| \leq \delta,$$

on $x \in \partial \Omega$,

then there exists a non-negative solution $F_s = M\mu + \sqrt{\mu}f_s \geq 0$ with $\int \int_{\Omega \times \mathbb{R}^3} f_s \sqrt{\mu} = 0$ to the problem

$$v \cdot \nabla_x F_s = Q(F_s, F_s), \quad F_s|_{\gamma_-} = \mu^\theta \int_{\gamma_+} F_s d\gamma,$$

such that, for all $0 \leq \zeta < \frac{1}{4+2\delta}$, $\beta > 4$,

$$\|\langle v \rangle^\beta e^{\zeta|v|^2} f_s\|_{\infty} + |\langle v \rangle^\beta e^{\zeta|v|^2} f_s|_{\infty} \lesssim \delta.$$ 

If $M\mu + \sqrt{\mu}g_s$ is an another solution with $\int \int_{\Omega \times \mathbb{R}^3} g_s \sqrt{\mu} = 0$ such that, for $\beta > 4$

$$\|\langle v \rangle^\beta g_s\|_{\infty} + |\langle v \rangle^\beta g_s|_{\infty} \ll 1,$$

then $f_s \equiv g_s$. 

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Fourier Law and Non-Isothermal Boundary
If $\theta(x)$ is continuous on $\partial \Omega$ then $F_s$ is continuous away from $\mathcal{D}$. In particular, if $\Omega$ is convex then $\mathcal{D} = \gamma_0$. On the other hand, if $\Omega$ is not convex then we can construct a continuous function $\theta(x)$ on $\partial \Omega$ such that the corresponding solution $F_s$ is not continuous.
Theorem: Dynamical Stability

Let $0 \leq \zeta < \frac{1}{4+2\delta}$, $\beta > 4$. There exists $\varepsilon_0 > 0$, depends on $\delta_0$, and $\lambda > 0$ such that if

$$\|\langle v \rangle^\beta e^{\zeta|v|^2} [f(0) - f_s]\|_\infty \leq \varepsilon_0$$

then there exists a unique non-negative dynamic solution

$$F(t) = \mu + f_s\sqrt{\mu} + f(t)\sqrt{\mu} \geq 0$$

to the dynamical problem

$$\partial_t F + v \cdot \nabla_x F = Q(F, F), \quad F(x, v) = \mu^\theta \int_{n(x) \cdot v > 0} F n \cdot v$$

for $x \in \partial\Omega$ and $n(x) \cdot v < 0$ such that

$$\|\langle v \rangle^\beta e^{\zeta|v|^2} [f(t) - f_s]\|_\infty \lesssim e^{-\lambda t}\|\langle v \rangle^\beta e^{\zeta|v|^2} [f(0) - f_s]\|_\infty$$
Why $\delta$—Expansion?

Fourier Law: a relation between the temperature and the heat flux

$$q_s = -\kappa(\theta_s) \partial_x \theta_s$$

for suitable positive smooth function $\kappa$.

Let $F_s$ be the solution to the steady Boltzmann equation

$$\theta_s(x) = \frac{1}{3\rho_s} \int_{\mathbb{R}^3} |v - u_s|^2 F_s(x, v) dv$$

$$u_s(x) = \frac{1}{\rho_s} \int_{\mathbb{R}^3} v F_s(x, v) dv$$

$$\rho_s(x) = \int_{\mathbb{R}^3} F_s(x, v) dv$$

$$q_s(x) = \frac{1}{2} \int_{\mathbb{R}^3} (v - u_s(x)) |v - u(x)|^2 F_s(x, v) dv.$$ 

Purpose: See the first order characterization of $F_s$.
What is $\delta-$Expansion? : $\mu_\delta-$Expansion

Wall Temperature

$$\theta(x) = \theta_0 + \delta \vartheta(x), \quad |\vartheta(x)| \leq 1, \quad x \in \partial \Omega.$$  

Wall Maxwellian

$$\mu_\delta(x, v) = \frac{1}{2\pi[\theta_0 + \delta \vartheta(x)]^2} \exp \left( - \frac{|v|^2}{2[\theta_0 + \delta \vartheta(x)]} \right)$$

Taylor Expansion in $\delta$ ($\mu_\delta$ is analytic in $\delta$)

$$\mu_\delta = \mu + \delta \mu_1 + \delta^2 \mu_2 + \cdots + \delta^m \mu_m + \cdots$$
What is $\delta-$Expansion? : $f_s \sim \delta f_1 + \delta^2 f_2 + \cdots$

Formal Expansion:

$$F_s = \mu + \sqrt{\mu}\left\{\delta f_1 + \delta^2 f_2 + \cdots\right\}$$

$$f_s = \delta f_1 + \delta^2 f_2 + \cdots$$

Plug in

$$\nu \cdot \nabla_x F_s = Q(F_s, F_s)$$

with Diffusive Boundary Condition to get the linear equation for $f_i$ (comparing the coefficients of power of $\delta$)

Once we solve $f_i$, define the Remainder $f^\delta_m$ such that

$$f_s = \delta f_1 + \delta^2 f_2 + \cdots + \delta^m f^\delta_m$$
Theorem: \( \delta \)-Expansion

\( \delta \)-Expansion is valid!

There exist \( f_1, f_2, \cdots, f_{m-1} \) and for all \( i = 1, 2, \cdots m - 1 \)

\[
\| \langle v \rangle^\beta e^{\zeta|v|^2} f_i \|_\infty \lesssim 1
\]

for all \( 0 \leq \zeta < \frac{1}{4}, \beta > 4 \)

and the remainder \( f_m^\delta \) exits and

\[
\| \langle v \rangle^\beta e^{\zeta|v|^2} f_m^\delta \|_\infty \lesssim 1
\]

for all \( 0 \leq \zeta < \frac{1}{4+2\delta}, \beta > 4 \)
Theorem : Criterion for Fourier Law

Let \( \Omega = [0, 1] \). If the Fourier Law holds for \( F_s = \mu + \sqrt{\mu}f_s \),

\[
F_s = \mu + \delta f_1 \sqrt{\mu} + O(\delta^2) \sqrt{\mu} \\
\theta_s = \theta_0 + \delta \theta_1 + O(\delta^2)
\]

then

\( \theta_1(x) \) is a linear function on \([0, 1]\)
From an available numeric simulation (Ohwada, Aoki, Sone, 1989) $\theta_1$ is not linear!

$\Downarrow$

Fourier Law is not valid at the kinetic regime!
Hydrodynamic Part : \( Pf \)

Linearized Boltzmann operator

- \( Lf = -\frac{1}{\sqrt{\mu}} [Q(\mu, \sqrt{\mu} f) + Q(\sqrt{\mu} f, \mu)] = \nu(v) f - Kf \)
- semi-positive : \( \langle Lg, g \rangle \gtrsim \| \{ I - P \} g \|^2_\nu \)
- kernel = ‘hydrodynamic part’

\[ Pf \equiv \left\{ a_g(t, x) + v \cdot b_g(t, x) + \frac{|v|^2 - 3}{2} c_g(t, x) \right\} \sqrt{\mu} \]

Boltzmann equation \( \implies \) macroscopic equation for \( b_f \)

\[ \Delta_x b_f = \partial_x^2 \{ I - P \} f + \cdots \]

ellipticity in \( H^k \) \( \implies \) Guo:VMB(Invent.Math.2003), VPL(JAMS2011); Gressman-Strain:BE without angular cut-off(JAMS2011)
Difficulties with boundary conditions

\[ \mathbf{P} f \equiv \left\{ a_f(t, x) + \mathbf{v} \cdot b_f(t, x) + \frac{|\mathbf{v}|^2 - 3}{2} c_f(t, x) \right\} \sqrt{\mu} \]

- \( \mathbf{P} f \) and \( \{ \mathbf{I} - \mathbf{P} \} f \) do not make sense at the boundary
- no boundary condition for \( a_f, c_f \), only \( b_f \cdot n(x) = 0 \) on \( \partial \Omega \)
Y. Guo: Initial Boundary Value Problem of BE, ARMA 2010

- $L^2$ Posivity: We Need A New Method!
- $L^\infty$ Bound: We Need A New Method!
New $L^2$ Positivity Estimate

\[ \mathbf{v} \cdot \nabla_x f + Lf = g, \quad f_{\gamma-} = P_\gamma f + r \]

with \[ \int_{\Omega \times \mathbb{R}^3} f \sqrt{\mu} = 0 = \int_{\Omega \times \mathbb{R}^3} g \sqrt{\mu} = \int_{n \cdot \mathbf{v} < 0} r \]

\[ \Rightarrow \| Pf \|_\nu \leq M \{ \| (I - P)f \|_\nu + \| (1 - P_\gamma)f \|_{2,+} \} + \cdots \]

\[ \text{Good Terms}! \]

- weak formulation (Green’s identity) + test functions
- constructive estimate with an explicit $M$
- dimension of $\Omega = 1, 2, 3$
New $L^2$ Positivity Estimate

Weak formulation (Green’s identity)

$$\int_\gamma \psi f - \int_\Omega \nu \cdot \nabla \psi f = - \int_\Omega \psi (I - P) f + \int_\Omega \psi g$$

bulk \hspace{1cm} f = \{a f + \nu \cdot b f + \frac{v^2 - 3}{2} c f\} \sqrt{\mu} + (I - P) f

boundary \hspace{1cm} f_\gamma = P_\gamma f + (1 - P_\gamma) f 1_{\gamma+} + r 1_{\gamma-}
Test functions

- for $c_f$:
  
  \[ \psi_c = (|v|^2 - \beta_c)\sqrt{\mu}\{v \cdot \nabla_x\}(-\Delta_0)^{-1}c_f \]
  
  with $\int_{\mathbb{R}^3} (|v|^2 - \beta_c)v_i^2 \mu(v)dv = 0$

- for $b_f$:
  
  \[ \psi_b^{i,j} = (v_i^2 - \beta_b)\sqrt{\mu}\partial_j(-\Delta_0)^{-1}(b_f)_j \]
  
  for all $i, j = 1, 2, \cdots d$
  
  with $\int_{\mathbb{R}^3} (v_i^2 - \beta_b)\mu(v)dv = 0$, for all $i$

  \[ \phi_b^{i,j} = v_i v_j |v|^2 \sqrt{\mu}\partial_j(-\Delta_0)^{-1}(b_f)_i \]
  
  for all $i \neq j$

- for $a_f$:
  
  \[ \psi_a = (|v|^2 - \beta_a)\{v \cdot \nabla_x\}(-\Delta_N)^{-1}a_f \]
  
  with $\int_{\mathbb{R}^3} (|v|^2 - \beta_a)(\frac{|v|^2}{2} - \frac{3}{2})(v_i)^2 \mu(v)dv = 0$ for all $i$
Future

Thanks !